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Chiral fermions on the lattice and index relations

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Abstract

Comparing recent lattice results on chiral fermions and old continuum results for the index puzzling questions arise. To clarify this issue we start with a critical reconsideration of the results on finite lattices. We then work out various aspects of the continuum limit. After determining bounds and norm convergences we obtain the limit of the anomaly term. Collecting our results the index relation of the quantized theory gets established. We then compare in detail with the Atiyah-Singer theorem. Finally we analyze conventional continuum approaches.

1. Introduction

Considering Ginsparg-Wilson (GW) fermions [1] recently an index theorem on the finite lattice has been formulated [2, 3]. An explicite form of the massless Dirac operator has been found in [4] from the overlap formalism [5]. For the eigenvalue flows within the latter an exact treatment has been presented [6]. The continuum limit of the anomaly term has been performed for the overlap Dirac operator [7, 8] along lines given for the Wilson-Dirac operator [9, 10] before. Altogether considerable progress in various directions has been initiated by these developments.

Comparing lattice results on the index with old continuum results [11, 12, 13] puzzling questions arise, which has recently led to concern about the relation $\mathrm{Tr}\,\gamma_5 = 0$ [14]. These questions are related to the observation [15] that with the simple GW relation and γ_5-hermiticity the Dirac operator gets normal and the index and the corresponding difference at the second real eigenvalue must add up to zero. On the other hand, no such restriction is known with the Atiyah-Singer index theorem [16, 17]. Since index relations are of fundamental importance it is highly desirable to clarify this issue. To do this is the main aim of the present paper.

To settle the indicated issue one obviously has to care about mathematical details. To have a reliable basis first a reconsideration of the situation on finite lattices is needed. With respect to the limit also the impact of a continous spectrum and the precise conditions on the lattice gauge fields remain to be considered. In case of the limit of the anomaly term we feel that, though mathematical questions have been addressed in [7], work remains still to be done and a different approach should be used.

We take advantage of the fact that the problem can be dealt with considering the theory in a background gauge field. This allows us to work in a unitary space on the finite lattice and in Hilbert space in the limit. To study the limit we find norm convergence a powerful tool. In its application we use various site-diagonal operators to get bounds – similarly as done for the square of the hermitean Wilson-Dirac operator H on the finite lattice in [18]. In addition we exploit orderings of selfadjoint operators – which generalizes the getting of lower bounds on H^2 on the finite lattice in [18, 19].

Having the index relation in the nonperturbative quantized theory established we consider the the Atiyah-Singer framework and work out the details needed for a comparison. This, in particular, concerns the main point of different cardinalities of the chiral subspaces in that framework. Finally we analyze conventional continuum approaches on the basis of our results.

In Section 2, starting with general Ward identities we see that with a background gauge field the operator relations to be investigated are the same ones for all expectation values. We introduce a family of alternative chiral transformations which lead to the same Ward identity as the usual one, however, allow to transport terms from the action contribution to that of the integration measure. This will allow us to discuss tranformations [3, 20] recently introduced in connection with the GW relation as well as an old claim [13] that the anomaly would arise from the measure.

In Section 3, to investigate the necessary properties on the finite lattice we start from the resolvent of the Dirac operator D and show that to get rid of eigennilpotents as well as to obtain chirality of eigenprojections in addition to γ_5-hermiticity we need normality of D. With these properties of D we then obtain general rules for real modes. Applying them to the global chiral Ward identity we get the sum rule of chiral differences, the respective difference at eigenvalue zero of D being the index[1]. It turns out that this sum rule – since one has to allow for a nonvanishing index – generally puts severe restrictions on the spectrum of D. To impose appropriate conditions we use a particular decomposition of D and give a general expression for one-dimensional spectral constraints.

[1]This is the usual definition on the lattice. By the mathematical definition the index is that of the Weyl operator associated to the continuum Dirac operator (while the index of the latter is zero).

In Section 4, considering consequences for the GW relation, we show that its general form does not guarantee normality of D and that the index relation claimed in [2] does not hold for it. We emphasize that one has to restrict to the simple form of the GW relation and that this form is actually a spectral constraint. Further, in that case the alternative transformation which transports the anomaly term to the measure contribution is seen to get that of [3]; we stress that with proper zero-mode regularization there is still the index term in the action contribution and thus the action noninvariant.

In Section 5 we first specify the details of the continuum limit and then address specific problems which remain to be settled. Since in the limit one can also get a continuous spectrum of D we study the respective changes in the spectral representations and investigate their consequences for the index relations. We also work out the issues related to the relation $\mathrm{Tr}\,\gamma_5 = 0$ in Hilbert space. We further determine the conditions on the lattice gauge fields needed to perform the limit. We handle subtleties of the gauge-field limit, frequently hidden in the notation, by properly distinguishing representations and using the isomorphism between the respective spaces.

In Section 6 we determine the limit of the anomaly term using norm convergences and getting bounds in the ways sketched above. Then combining with results of preceding sections we arrive at the index theorem as it holds in nonperturbative quantized theory.

In Section 7 we consider the Atiyah-Singer theorem and derive relations needed for a detailed comparison. We then discuss the differences to the relation obtained in nonperturbative quantized theory and stress the fundamental one of different cardinalities of the chiral subspaces in the Atiah-Singer case.

In Section 8, analyzing conventional continuum approaches, we show that the Pauli-Villars approach as well as the path-integral approach neither conform with the nonperturbative quantized theory nor with the Atiyah-Singer framework but – as perturbation theory – rely on a modification of the theory at the level of the Ward identity.

In Section 9 we collect conclusions.

2. Ward identities

2.1 General form of identity

Fermionic Ward identities arise from the condition that $\int[\mathrm{d}\bar{\psi}\mathrm{d}\psi]\mathrm{e}^{-S_\mathrm{f}}\mathcal{O}$ must not change under a transformation of the integration variables. Considering the transformation

$$\psi' = \exp(i\eta\Gamma)\psi \quad , \quad \bar{\psi}' = \bar{\psi}\exp(i\eta\bar{\Gamma}) \tag{2.1}$$

with parameter η this can be expressed by the identity

$$\frac{\mathrm{d}}{\mathrm{d}\,\eta}\int[\mathrm{d}\bar{\psi}'\mathrm{d}\psi']\mathrm{e}^{-S_\mathrm{f}'}\mathcal{O}'\Big|_{\eta=0} = 0 \tag{2.2}$$

where $S'_{\rm f} = \bar\psi' M \psi'$. Evaluation of (2.2) using the rules for Grassmann variables gives

$$i\int[\mathrm{d}\bar\psi\mathrm{d}\psi]\mathrm{e}^{-S_{\rm f}}\Big(-\,\mathrm{Tr}(\bar\Gamma+\Gamma)\mathcal{O}-\bar\psi(\bar\Gamma M+M\Gamma)\psi\mathcal{O}+\bar\psi\bar\Gamma\frac{\partial\mathcal{O}}{\partial\bar\psi}-\frac{\partial\mathcal{O}}{\partial\psi}\Gamma\psi\Big)=0 \tag{2.3}$$

(with $\frac{\partial\mathcal{O}}{\partial\psi}\Gamma\psi \equiv \sum_l \frac{\partial\mathcal{O}}{\partial\psi_l}(\Gamma\psi)_l$). The three contributions in (2.3) stem from the derivative of the integration measure, from that of the action, and from that of $\mathcal{O}$, respectively.

In the present context one usually puts $\mathcal{O}=1$. We can, however, do better integrating out the ψ and $\bar\psi$ fields in the second term of (2.3). For this purpose we use the identity

$$\begin{aligned} 0 &= \frac{1}{2}\int[\mathrm{d}\bar\psi\mathrm{d}\psi]\bigg(\Big(\frac{\partial}{\partial\psi}M^{-1}\Big)_j(\mathrm{e}^{-S_{\rm f}}\psi_k\mathcal{O})+\Big(M^{-1}\frac{\partial}{\partial\bar\psi}\Big)_k(\mathrm{e}^{-S_{\rm f}}\bar\psi_j\mathcal{O})\bigg) \\ &= \int[\mathrm{d}\bar\psi\mathrm{d}\psi]\mathrm{e}^{-S_{\rm f}}\bigg(\bar\psi_j\psi_k\mathcal{O}+M^{-1}_{kj}\mathcal{O}+\frac{1}{2}\Big(\frac{\partial\mathcal{O}}{\partial\psi}M^{-1}\Big)_j\psi_k-\frac{1}{2}\bar\psi_j\Big(M^{-1}\frac{\partial\mathcal{O}}{\partial\bar\psi}\Big)_k\bigg) \end{aligned} \tag{2.4}$$

which follows from the fact that $\int[\mathrm{d}\bar\psi\mathrm{d}\psi](\partial/\partial\psi_l)F=0$ and $\int[\mathrm{d}\bar\psi\mathrm{d}\psi](\partial/\partial\bar\psi_l)F=0$ for any function F. Then (2.3) becomes

$$\begin{aligned} &i\mathrm{Tr}\Big(-\bar\Gamma-\Gamma+M^{-1}(\bar\Gamma M+M\Gamma)\Big)\int[\mathrm{d}\bar\psi\mathrm{d}\psi]\mathrm{e}^{-S_{\rm f}}\mathcal{O} \\ &+\frac{i}{2}\int[\mathrm{d}\bar\psi\mathrm{d}\psi]\mathrm{e}^{-S_{\rm f}}\Big(\frac{\partial\mathcal{O}}{\partial\psi}M^{-1}R\psi+\bar\psi RM^{-1}\frac{\partial\mathcal{O}}{\partial\bar\psi}\Big)=0 \end{aligned} \tag{2.5}$$

where $R=\bar\Gamma M-M\Gamma$. To evaluate the derivatives of $\mathcal{O}$ in (2.5) we note that the fermionic part of $\mathcal{O}$ with nonzero contribution to the integral generally is a linear combination of products of type $\mathcal{P}=\psi_{j_1}\bar\psi_{k_1}\dots\psi_{j_s}\bar\psi_{k_s}$ for which by

$$\int[\mathrm{d}\bar\psi\mathrm{d}\psi]\mathrm{e}^{-S_{\rm f}}\psi_{j_1}\bar\psi_{k_1}\dots\psi_{j_s}\bar\psi_{k_s}=\sum_{l_1\dots l_s}\epsilon^{k_1\dots k_s}_{l_1\dots l_s}M^{-1}_{j_1l_1}\dots M^{-1}_{j_sl_s}\det M \tag{2.6}$$

(with $\epsilon^{k_1\dots k_s}_{l_1\dots l_s}=+1$, -1, or 0 if $k_1\dots k_s$ even, odd, or no permutation of $l_1\dots l_s$) we find

$$\begin{aligned} &-\int[\mathrm{d}\bar\psi\mathrm{d}\psi]\mathrm{e}^{-S_{\rm f}}\frac{\partial\mathcal{P}}{\partial\psi}M^{-1}R\psi=+\int[\mathrm{d}\bar\psi\mathrm{d}\psi]\mathrm{e}^{-S_{\rm f}}\bar\psi RM^{-1}\frac{\partial\mathcal{P}}{\partial\bar\psi}= \\ &\sum_{\mathrm{pos}\,z}\sum_{l_1\dots l_s}\epsilon^{k_1\dots k_s}_{l_1\dots l_s}M^{-1}_{j_1l_1}\dots M^{-1}_{j_{z-1}l_{z-1}}(M^{-1}RM^{-1})_{j_zl_z}M^{-1}_{j_{z+1}l_{z+1}}M^{-1}_{j_sl_s}\det M \end{aligned} \tag{2.7}$$

where the first sum is over all positions of the $M^{-1}RM^{-1}$ factor in the product of the M^{-1}. This shows that the terms in (2.5) with derivatives of $\mathcal{O}$ cancel and we remain with

$$i\ \mathrm{Tr}\Big(-\bar\Gamma-\Gamma+M^{-1}(\bar\Gamma M+M\Gamma)\Big)\int[\mathrm{d}\bar\psi\mathrm{d}\psi]\mathrm{e}^{-S_{\rm f}}\mathcal{O}=0\ . \tag{2.8}$$

From (2.8) it is seen that in a background gauge field the expectation values factorize so that also for arbitrary $\mathcal{O}$ (and not only for $\mathcal{O}=1$) it suffices to consider the identity

$$\frac{1}{2}\mathrm{Tr}\Big(-\bar\Gamma-\Gamma+M^{-1}(\bar\Gamma M+M\Gamma)\Big)=0 \tag{2.9}$$

where $-\frac{1}{2}\mathrm{Tr}(\bar\Gamma+\Gamma)$ is the measure contribution and $\frac{1}{2}\mathrm{Tr}\Big(M^{-1}(\bar\Gamma M+M\Gamma)\Big)$ the action contribution.

2.2 Chiral transformations

For the global chiral transformation, in which case one has[2]

$$\Gamma = \bar{\Gamma} = \gamma_5 \ , \tag{2.10}$$

the measure contribution vanishes and (2.9) becomes

$$\frac{1}{2}\mathrm{Tr}(M^{-1}\{\gamma_5, M\}) = 0 \ . \tag{2.11}$$

Obviously this can also be read as $\mathrm{Tr}\,\gamma_5 = 0$, of which the Ward identity is the particular decomposition which is dictated by the chiral transformation.

In order that the Ward identity makes sense we have to care about the existence of M^{-1}. To deal with zero modes of a Dirac operator D we therefore put $M = D - \zeta$ with the parameter ζ being in the resolvent set (i.e. not in the spectrum of D) and let ζ go to zero only in the final result. Thus from (2.11) we altogether get

$$\mathrm{Tr}\,\gamma_5 = \frac{1}{2}\mathrm{Tr}\Big((D-\zeta)^{-1}\{\gamma_5, D\}\Big) - \zeta\,\mathrm{Tr}\Big((D-\zeta)^{-1}\gamma_5\Big) = 0 \ . \tag{2.12}$$

To have definite names in our discussions we shall call the first term in (2.12) anomaly term and the second one index term.

To get the local chiral transformation one has to use

$$\Gamma = \bar{\Gamma} = \gamma_5 \hat{e}(n) \tag{2.13}$$

where $\hat{e}(n)$ is a projection which in lattice-space representation is given by

$$\Big(\hat{e}(n)\Big)_{n''n'} = \delta^4_{n''n}\delta^4_{nn'} \ . \tag{2.14}$$

Then the identity (2.9) becomes $\frac{1}{2}\mathrm{Tr}\Big(M^{-1}\{\gamma_5\hat{e}(n), M\}\Big) = 0$, which may also be read as $\mathrm{Tr}\Big(\gamma_5\hat{e}(n)\Big) = 0$. Decomposing $\{\gamma_5\hat{e}(n), M\}$ by $M = \frac{1}{2}(M - \gamma_5 M \gamma_5) + \frac{1}{2}(M + \gamma_5 M \gamma_5)$ into parts anticommuting and commuting with γ_5 and inserting $M = D - \zeta$ one now has

$$\begin{aligned} \mathrm{Tr}\Big(\gamma_5\hat{e}(n)\Big) \ &= \ \frac{1}{4}\mathrm{Tr}\Big(M^{-1}[\hat{e}(n), [\gamma_5, D]]\Big) + \\ & \quad\ \frac{1}{4}\mathrm{Tr}\Big(M^{-1}\{\hat{e}(n), \{\gamma_5, D\}\}\Big) - \zeta\,\mathrm{Tr}\Big(M^{-1}\gamma_5\hat{e}(n)\Big) = 0 \ . \end{aligned} \tag{2.15}$$

The first term in (2.15) is seen to vanish upon summation over n and accordingly corresponds to the divergence of the singlet axial vector current. The second term and the third term in (2.15) are the local versions of the anomaly term and of the index term, respectively.

[2]For simplicity we write γ_5 instead of $\gamma_5 \bigotimes 1\!\!1_s$ wherever this cannot cause misunderstandings.

2.3 Alternative chiral transformations

We introduce a family of alternative global chiral transformation by

$$\Gamma = \gamma_5 - K \quad , \quad \bar{\Gamma} = \gamma_5 - \bar{K} \tag{2.16}$$

which inserted into (2.9) gives

$$-\frac{1}{2}\mathrm{Tr}(\bar{\Gamma} + \Gamma) = +\frac{1}{2}\mathrm{Tr}(K + \bar{K}) \tag{2.17}$$

for the measure contribution and

$$\frac{1}{2}\mathrm{Tr}\Big(M^{-1}(\bar{\Gamma}M + M\Gamma)\Big) = \frac{1}{2}\mathrm{Tr}(M^{-1}\{\gamma_5, M\}) - \frac{1}{2}\mathrm{Tr}(K + \bar{K}) \tag{2.18}$$

for the action contribution. Obviously the extra term of the latter cancels the measure term so that again the result (2.11) is obtained for any operators K and $\bar{K}$.

While the Ward identity remains the same for these transformations, they may be used to change the origin of its terms. For example, with

$$K = \frac{1}{2}M^{-1}\{\gamma_5, D\} \quad , \quad \bar{K} = \frac{1}{2}\{\gamma_5, D\}M^{-1} \tag{2.19}$$

the anomaly term of (2.12) is transported from the action contribution to the measure contribution. Similarly by

$$K = \frac{1}{2}M^{-1}\{\gamma_5, M\} \quad , \quad \bar{K} = \frac{1}{2}\{\gamma_5, M\}M^{-1} \tag{2.20}$$

both terms of (2.12) are transported to the measure contribution. In case of (2.20) the action of form $\bar{\psi}M\psi$ is invariant under the transformation.

To get the local versions of the alternative chiral transformations one simply has to replace γ_5 of the global cases by $\gamma_5\hat{e}(n)$. With these transformations again the usual result (2.15) is obtained.

3. Chiral properties

3.1 Mathematical framework

In numerical work the quantities we are dealing with are given in matrix representation. Here we have to care about their mathematical meaning. The outcome of the Grassmann integrals can be expressed by determinants, minors, and generalizations thereof, and in turn by traces of powers of the operators [21]. The definition of such expressions requires the mapping to be within the respective space itself. In addition, to have independence of the particular basis one must restrict basis transformations to similarity transformations as is done in a vector space.

Thus the theoretical understanding of the quantities in (2.9) must be that of operators in a vector space (with dimension equal to number of sites times spinor dimension times gauge-group dimension) which map to this vector space itself. The trace then depends solely on the particular operators. At the same time the formulation of eigenvalue problems – also important in the present context – becomes possible.

We need, however, still more. To be able to define adjoint operators we, in addition, must introduce an inner product. This means that the vector space gets a unitary one. Then basis transformations are restricted to unitary ones and one can define normality, including hermiticity and unitarity, and also γ_5-hermiticity of operators (while triangulations of matrices and transformation to the Jordan form are no longer generally possible). We further note that then gauge transformations can be considered as a particular class of basis transformations.

3.2 Basic relations

A general operator D has the spectral representation $D = \sum_j (\lambda_j P_j + Q_j)$ with eigenprojections P_j and eigennilpotents Q_j. Its resolvent is meromorphic, regular at infinity, and given by [22]

$$(D-\zeta)^{-1} = -\sum_{j=1}^{s}\Big((\zeta-\lambda_j)^{-1}P_j + \sum_{k=1}^{d_j-1}(\zeta-\lambda_j)^{-k-1}Q_j^k\Big) \tag{3.1}$$

where $d_j = \mathrm{Tr}\, P_j$ is the dimension of the subspace onto which P_j projects. For P_j and Q_j one has $P_jP_l = \delta_{jl}P_j$, $P_jQ_l = Q_lP_j = \delta_{jl}Q_j$, $Q_jQ_l = 0$ for $j \neq l$, and $Q_j^{d_j} = 0$.

Using hermitean γ-matrices we require that for the Dirac operator $D^\dagger = \gamma_5 D\gamma_5$ holds, i.e. that it is γ_5-hermitean[3]. Then the resolvent satisfies

$$(D-\zeta)^{-1} = \gamma_5(D^\dagger-\zeta)^{-1}\gamma_5 \,. \tag{3.2}$$

Expressing $(D-\zeta)^{-1}$ and $(D^\dagger-\zeta)^{-1}$ in (3.2) by (3.1) and integrating over ζ around a circle enclosing only one eigenvalue λ_j we obtain

$$P_j = \gamma_5 P_j^\dagger \gamma_5 \quad \text{for} \quad \lambda_j \quad \text{real} \tag{3.3}$$

and find that for each complex λ_j with P_j there also occurs a value λ_k with P_k where $\lambda_k = \lambda_j^*$ and $P_k = \gamma_5 P_j^\dagger \gamma_5$.

[3]With anti-hermitean γ-matrices instead anti-γ_5-hermiticity $\gamma_5 D\gamma_5 = -D^\dagger$ is to be required. Our real-mode rules then change into purely-imaginary-mode rules.

To get chiral properties it is necessary that D and γ_5 have simultaneous eigenvectors at least for $\lambda_j = 0$, which means that P_j should commute with γ_5. From (3.3) we see that

$$[\gamma_5, P_j] = 0 \quad \text{iff} \quad P_j^\dagger = P_j \quad \text{for} \quad \lambda_j \quad \text{real} \tag{3.4}$$

so that we should have $P_j^\dagger = P_j$ for this. However, one cannot specify such condition itself because P_j would be only available after the eigenvalue problem is solved. Therefore, one must find a condition on D which generally implies the respective property of P_j. The condition appropriate here is $[D, D^\dagger] = 0$, i.e. that D is normal. Then one gets $P_j^\dagger = P_j$ for all j. At the same time it follows that $Q_j = 0$ for all j so that the spectral representation simplifies to $D = \sum_j \lambda_j P_j$.

3.3 Real-mode rules

With γ_5-hermiticity and normality of D we can write its spectral representation in the form

$$D = \sum_{j\ (\mathrm{Im}\lambda_j=0)} \lambda_j P_j^{(5)} + \sum_{k\ (\mathrm{Im}\lambda_k>0)} (\lambda_k P_k^{(1)} + \lambda_k^* P_k^{(2)}) \tag{3.5}$$

in which the notation of the projections is such that they satisfy

$$\gamma_5 P_j^{(5)} \gamma_5 = P_j^{(5)} \quad , \quad \gamma_5 P_j^{(1)} \gamma_5 = P_j^{(2)} \ . \tag{3.6}$$

Because of $[\gamma_5, P_j^{(5)}] = 0$ we get $P_j^{(5)} = P_j^{(+)} + P_j^{(-)}$ with $\gamma_5 P_j^{(\pm)} = \pm P_j^{(\pm)}$ so that

$$\mathrm{Tr}(\gamma_5 P_j^{(5)}) = d_j^{(+)} - d_j^{(-)} \tag{3.7}$$

where $d_j^{(\pm)} = \mathrm{Tr}\, P_j^{(\pm)}$ is the dimension of the subspace onto which $P_j^{(\pm)}$ projects. From (3.6) according to $\mathrm{Tr}(\gamma_5 P_j^{(1)}) = \mathrm{Tr}(\gamma_5 P_j^{(1)\,2}) = \mathrm{Tr}(\gamma_5 P_j^{(1)} P_j^{(2)})$ we also have

$$\mathrm{Tr}(\gamma_5 P_j^{(1)}) = \mathrm{Tr}(\gamma_5 P_j^{(2)}) = 0 \ . \tag{3.8}$$

Since with (3.5) we get for any function $g(D)$

$$g(D) = \sum_{j\ (\mathrm{Im}\lambda_j=0)} g(\lambda_j) P_j^{(5)} + \sum_{k\ (\mathrm{Im}\lambda_k>0)} \Big(g(\lambda_k) P_k^{(1)} + g(\lambda_k^*) P_k^{(2)} \Big) \tag{3.9}$$

using (3.7) and (3.8) we obtain

$$\mathrm{Tr}(\gamma_5 g(D)) = \sum_{\lambda_j \mathrm{real}} g(\lambda_j) I(\lambda_j) \tag{3.10}$$

where we have introduced

$$I(\lambda_j) = d_j^{(+)} - d_j^{(-)} \quad \text{for} \quad \lambda_j \quad \text{real} \ . \tag{3.11}$$

We now apply (3.10) to the global chiral Ward identity (2.12). For the index term and the anomaly term we find

$$-\lim_{\zeta\to 0} \mathrm{Tr}\Big((D-\zeta)^{-1}\gamma_5\,\zeta\Big) = I(0)\ , \tag{3.12}$$

$$\lim_{\zeta\to 0} \frac{1}{2}\mathrm{Tr}\Big((D-\zeta)^{-1}\{\gamma_5, D\}\Big) = \sum_{\lambda_j\neq 0\ \text{real}} I(\lambda_j)\ , \tag{3.13}$$

respectively, so that for $\zeta \to 0$ the identity (2.12) gets the sum rule for real modes

$$\mathrm{Tr}(\gamma_5) = \sum_{\lambda_j\ \text{real}} I(\lambda_j) = 0\ . \tag{3.14}$$

From (3.14) it becomes obvious that one has the same total number of right-handed and left-handed modes and that the mechanism leading to a nonvanishing index $I(0)$ works via compensating numbers of modes at different λ_j. Therefore the index can only be nonvanishing if a corresponding difference from nonzero eigenvalues exists. Thus in addition to 0, allowing for zero modes, there must be at least one further real value available in the spectrum. Obviously this puts severe restrictions on the spectrum of the Dirac operator D.

3.4 Spectral constraints

To study how spectral constraints accounting for the restrictions due to the sum rule (3.14) can be imposed on D we use the decomposition

$$D = u + iv \quad \text{with} \quad u = u^\dagger = \frac{1}{2}(D + D^\dagger) \quad , \quad v = v^\dagger = \frac{1}{2i}(D - D^\dagger)\ . \tag{3.15}$$

By the γ_5-hermiticity of D one then in addition has

$$u = \frac{1}{2}\gamma_5\{\gamma_5, D\} \quad , \quad v = \frac{1}{2}\gamma_5[\gamma_5, D] \tag{3.16}$$

from which $[\gamma_5, u] = 0$ and $\{\gamma_5, v\} = 0$ follow.

The crucial observation now is that the normality of D implies $[u, v] = 0$ so that for u, v, and D one gets simultaneous eigenvectors and the eigenvalues of u and v are simply the real and imaginary parts, respectively, of those of D. This opens the way to formulate appropriate constraints on the spectrum by imposing conditions on u and v.

In particular, we may restrict the spectrum to a one-dimensional set specifying some function $f(u, v)$ and requiring

$$f(u, v) = 0\ . \tag{3.17}$$

The function $f(x, y)$ considered for real x and y must be such that for $x = 0$ and for at least one further value $x = x_l$ the condition $f(x, y) = 0$ implies $y = 0$. Since complex eigenvalues come in pairs we also must have $f(x,-y)=f(x,y)$. Requiring that there are r such values x_l we can satisfy these conditions by choosing the general form

$$f(u,v)=u\,(u-x_1)\dots(u-x_r)\,g(u,v^2)+v^2\,h(u,v^2) \tag{3.18}$$

where $g(u,0)\neq 0$, $h(0,v^2)\neq 0$, and $h(x_l,v^2)\neq 0$. With (3.18) the constraint (3.17) then can be cast into the form of a condition on D itself by inserting (3.15) or (3.16).

A most simple example is obtained putting $r=g=h=1$ and $x_1=2\rho$ in (3.18). Inserting (3.16) this is seen to give just the GW-relation of form (4.4). Similarly the relation $D+\gamma_5 D\gamma_5=2a^{2k+1}(\gamma_5 D)^{2k+2}$ proposed in [23] follows from the choice $r=1$, $x_1=a^{-1}$, $g=2a\Big(1+au+(au)^2+\ldots+(au)^{2k}\Big)$, and $h=2a\sum_{l=0}^{k}\begin{pmatrix}k+1\\ l+1\end{pmatrix}(av)^{2l}(au)^{2(k-l)}$. In these special cases because of $r=1$ the sum rule (3.14) gets simply $I(0)+I(x_1)=0$. Apparently it is straightforward to construct further constraints along these lines.

4. Discussion of GW-related results

4.1 Form and meaning of GW-relation

The general GW relation [1] can be written as

$$\{\gamma_5,D\}=2D\gamma_5 RD \tag{4.1}$$

where R is a hermitean operator which is trivial in Dirac space. From (4.1) using γ_5-hermititcity of D and $[\gamma_5,R]=0$ one obtains $[D,D^\dagger]=2D^\dagger[R,D]D^\dagger$. Thus it is seen that one should have $[R,D]=0$ in order that D gets normal. To get this property generally one has to put R equal to a multiple of the identity.

To check what happens for general R we insert (4.1) into the identity (2.12) and find

$$\mathrm{Tr}\,\gamma_5=\mathrm{Tr}(\gamma_5 RD)-\zeta\,\mathrm{Tr}\Big(\gamma_5(D-\zeta)^{-1}\Big)+\zeta^2\,\mathrm{Tr}\Big(\gamma_5 R(D-\zeta)^{-1}\Big)=0\;. \tag{4.2}$$

Dividing (4.2) by ζ, expressing $(D-\zeta)^{-1}$ by (3.1), and integrating over ζ around a circle enclosing only the eigenvalue $\lambda_j=0$ we obtain

$$\mathrm{Tr}\,\gamma_5=\mathrm{Tr}(\gamma_5 RD)+\mathrm{Tr}\Big(\gamma_5(P_j+RQ_j)\Big)=0\quad\text{for}\quad\lambda_j=0\;. \tag{4.3}$$

We see now that the relation for the index claimed in [2] does not hold for general R. To get it firstly one should have $P_j^\dagger=P_j$ in (4.3) so that by (3.4) one would get $[\gamma_5,P_j]=0$ as is necessary for chirality. Secondly the term in (4.3) with the eigennilpotent Q_j should disappear. Both requirements would have been met if D would have been normal.

Putting $R = (2\rho)^{-1}\mathbb{1}$ in (4.1) to have normality of D we arrive at the simple GW relation

$$\{\gamma_5, D\} = \rho^{-1} D \gamma_5 D \ . \tag{4.4}$$

Requiring also γ_5-hermiticity of D, the condition (4.4) means that $\rho(D + D^\dagger) = DD^\dagger = D^\dagger D$ should hold, i.e. that $D/\rho - 1$ should be unitary. Thus the actual content of (4.4) is the restriction of the spectrum of D to the circle through zero with center at ρ. In fact, as already mentioned, it is simply a spectral constraint of type (3.17) with (3.18).

For $R = (2\rho)^{-1}\mathbb{1}$ the form (4.2) of the identity (2.12) becomes

$$\mathrm{Tr}\,\gamma_5 = (2\rho)^{-1}\mathrm{Tr}(\gamma_5 D) - \zeta\,\mathrm{Tr}\Big(\gamma_5 (D-\zeta)^{-1}\Big) + (2\rho)^{-1}\zeta^2\,\mathrm{Tr}\Big(\gamma_5 (D-\zeta)^{-1}\Big) = 0 \ . \tag{4.5}$$

In (4.5), because of $Q_j = 0$ for normal D, the term with ζ^2 does no longer contribute in the limit $\zeta \to 0$ so that one remains with the form

$$(2\rho)^{-1}\mathrm{Tr}(\gamma_5 D) \tag{4.6}$$

of the anomaly term, while the index term is the same as in (2.12). As for all constraints with $r = 1$ in (3.18) the sum rule (3.14) now has only two terms

$$\mathrm{Tr}\,\gamma_5 = I(0) + I(2\rho) = 0 \ . \tag{4.7}$$

With the special form (4.6) of the anomaly term one can also use the relation $\mathrm{Tr}(\gamma_5 D) = \sum_{\lambda_j \neq 0 \text{ real}} \lambda_j\, I(\lambda_j)$ following from (3.10) to obtain $(2\rho)^{-1}\mathrm{Tr}(\gamma_5 D) = I(2\rho)$.

4.2 Alternative transformations in GW case

For the alternative transformation with (2.19), which transports the anomaly term to the measure contribution, in case of the GW relation (4.4) with the form (4.6) of the anomaly term one gets

$$K = (2\rho)^{-1}\gamma_5 D \quad , \quad \bar{K} = (2\rho)^{-1} D \gamma_5 \ . \tag{4.8}$$

This gives the transformation introduced in infinitesimal form in [3] which by (4.4) leaves the classical action $\bar\psi D \psi$ invariant. With (4.8) the measure contribution gets $(2\rho)^{-1}\mathrm{Tr}(\gamma_5 D)$. However, there still remains the index term $-\zeta\mathrm{Tr}(\gamma_5(D-\zeta)^{-1})$ of the Ward identity in the action contribution.

The remaining action contribution is missing in [3] since no zero-mode regularization has been used. Thus it looks there like the action would also be invariant in the quantum case with zero modes, as is not correct. In an independent step, which without motivation uses a decomposition of $\mathrm{Tr}\,\gamma_5 = 0$ equivalent to (4.5), what should have been in the action contribution is calculated there from the measure term.

The generalized transformation of [20] is obtained by putting $K = \gamma_5 S D$, $\bar K = D T \gamma_5$ in (2.16) where S and T are hermitean operators which are trivial in Dirac space. If D satisfies the general GW relation (4.1) with $R = S + T$ the classical action $\bar\psi D \psi$ is invariant under this transformation.

5. Continuum limit

5.1 Steps towards the limit

The continuum limit is the final part of the nonperturbative definition of a quantum field theory provided by the lattice formulation. It appears worthwhile to remind of the general scheme and of the succession of the steps: (1) The quantization prescription is to discretize a continuum action such that the classical continuum limit, namely that of the action alone, gives the continuum expression. (2) With the discretized action the functional integrals are evaluated on the finite lattice; the lattice-spacing parameter drops out at this stage. (3) The infinite-volume limit is taken on the basisis of a sequence of finite-lattice results. (4) Letting the parameters approach a critical value the limit of zero lattice spacing is obtained.

Since for the present purpose it suffices to consider the quantum field theory of fermions in a background gauge field we consider here the details for this case. Obviously it remains to deal with Steps (3) and (4). By Step (3) our unitary space gets of infinite dimension. In order to be able to perform limits we must complete it to a separable Hilbert space. In the relations considered so far the main change is that the spectrum of the operator D then can include a continuum. Further the existence of traces is to be checked.

Step (4) is most familiar in theories with a bare mass which is parametrized as $m_{\mathrm{bar}}(a) = a\, m_{\mathrm{physical}}$ such that with decreasing lattice spacing a it goes to a critical singularity (where the correlation lenght diverges). In this way, given the n-representation of space, letting $an \to x$, one gets the x-representation.

In the present context Step (4) has to deal with the gauge field $U_{\mu n}$. The situation is similar to that for a theory with $m_{\mathrm{bar}}(a) \to 0$ in that we have to parametrize such that $U_{\mu n}(a) \to \mathbb{1}$ for $a \to 0$. However, unlike a bare mass the field $U_{\mu n}$ depends on n. This introduces a subtlety, frequently hidden in the notation, which is revealed carefully distinguishing n-representation and x-representation.

The mostly used relation of lattice gauge fields $U_{\mu n}$ to continuum gauge fields $A_\mu(x)$ then reads $U_{\mu n} = \mathcal{P} \exp\Big(\int_{an}^{a(n+\hat{\mu})} \mathrm{d}^4x\, A_\mu(x)\Big)$ where $\mathcal{P}$ denotes path ordering. Now $B_{\mu n}$ in $U_{\mu n} = \exp(iB_{\mu n})$ is a certain average over $aA_\mu(y)$ where $an_\nu \le y_\nu \le a(n_\nu + 1)$ and for small a one essentially has $B_{\mu n} \approx aA_\mu(an)$. In the limit requiring $an \to x$ for $a \to 0$ with fixed x implies that also $|n_\nu| \to \infty$. The latter not only needs Step (3) as a prerequisite but also causes the subtlety that this are only the $U_{\mu n}$ with large $|n_\nu|$ which enter.

Starting from the lattice the gauge fields are given there. Therefore we intend to find the precise conditions on the lattice gauge fields which are needed in order that the correct limit exists (which is in contrast to the frequent view of prescribing classical fields).

After taking the infinite-volume limit we are in n-representation which means that the abstract Hilbert space is realized as that of sequences. Alternatively we can realize it as that of square-integrable functions. This is what we have in x-representation which we prefer after taking the limit $a \to 0$ of vanishing lattice spacing. Since the space of sequences as well as the space of square-integrable functions both represent a separable Hilbert space they are isomorphic so that we can use both descriptions.

5.2 Spectrum in Hilbert space

In Hilbert space the spectrum of D can also get continuous parts. Using the decomposition (3.15) we then can rely on the spectral representations of the selfadjoint operators u and v in terms of operator Stieltjes integrals

$$D = \int \mathrm{d}\, E^{\mathrm{I}}_{\alpha}\, \alpha + i \int \mathrm{d}\, E^{\mathrm{II}}_{\beta}\, \beta\ . \tag{5.1}$$

Because D is normal the spectral families of u and v commute, $[E^{\mathrm{I}}_{\alpha}, E^{\mathrm{II}}_{\beta}] = 0$. From γ_5-hermiticity, inserting (5.1) into $D = \gamma_5 D^\dagger \gamma_5$, it follows that

$$\gamma_5 E^{\mathrm{I}}_{\alpha} \gamma_5 = E^{\mathrm{I}}_{\alpha} \quad , \quad \gamma_5 E^{\mathrm{II}}_{\beta} \gamma_5 = E^{\mathrm{II}}_{-\beta}\ . \tag{5.2}$$

Using $\int \mathrm{d}\, E^{\mathrm{I}}_{\alpha} = \mathbb{1}$ and $\int \mathrm{d}\, E^{\mathrm{II}}_{\beta} = \mathbb{1}$ we can write (5.1) as

$$D = \int \mathrm{d}\, E^{\mathrm{I}}_{\alpha} \int \mathrm{d}\, E^{\mathrm{II}}_{\beta}\, (\alpha + i\beta)\ . \tag{5.3}$$

In order to decompose this with respect to values with $\beta = 0$, $\beta > 0$, and $\beta < 0$ we define

$$E^{(5)}_{\alpha} = E^{\mathrm{I}}_{\alpha} \int_{-0}^{+0} \mathrm{d}\, E^{\mathrm{II}}_{\beta}\ , \quad E^{(1)}_{\beta} = \begin{cases} E^{\mathrm{II}}_{\beta} & \text{for } \beta > 0 \\ 0 & \text{otherwise} \end{cases} \quad , \quad E^{(2)}_{\beta} = \begin{cases} E^{\mathrm{II}}_{\beta} & \text{for } \beta < 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.4}$$

for which by (5.2)

$$\gamma_5 E^{(5)}_{\alpha} \gamma_5 = E^{(5)}_{\alpha} \quad , \quad \gamma_5 E^{(1)}_{\beta} \gamma_5 = E^{(2)}_{\beta} \tag{5.5}$$

holds. With (5.4) now (5.3) becomes

$$D = \int \mathrm{d}\, E^{(5)}_{\alpha} \alpha + \int \mathrm{d}\, E^{\mathrm{I}}_{\alpha} \int \mathrm{d}\, E^{(1)}_{\beta} (\alpha + i\beta) + \int \mathrm{d}\, E^{\mathrm{I}}_{\alpha} \int \mathrm{d}\, E^{(2)}_{\beta} (\alpha + i\beta) \tag{5.6}$$

which generalizes (3.5). For functions $g(D)$ we thus get

$$g(D) = \int \mathrm{d}\, E^{(5)}_{\alpha} g(\alpha) + \int \mathrm{d}\, E^{\mathrm{I}}_{\alpha} \int \mathrm{d}\, E^{(1)}_{\beta} g(\alpha + i\beta) + \int \mathrm{d}\, E^{\mathrm{I}}_{\alpha} \int \mathrm{d}\, E^{(2)}_{\beta} g(\alpha + i\beta) \tag{5.7}$$

where we require $g(\lambda)$ to be continuous to guarantee that the sum of the Stieltjes integrals still equals $g(D) = \int \mathrm{d}\, E^{\mathrm{I}}_{\alpha} \int \mathrm{d}\, E^{\mathrm{II}}_{\beta}\, g(\alpha + i\beta)$. With the relation (5.7) we now have the generalization of (3.9).

5.3 Modification of real-mode relations

With the spectral representation (5.7) $\mathrm{Tr}(\gamma_5 g(D))$ is defined in terms of the traces of the spectral family. Since by (5.2) one has $[\gamma_5, E_\alpha^{\mathrm{I}}] = 0$ and with (5.5) gets $\gamma_5 E_\beta^{(1)} = \gamma_5 E_\beta^{(1)\,2} = E_\beta^{(2)}\gamma_5 E_\beta^{(1)}$ and $\gamma_5 E_\beta^{(2)} = E_\beta^{(1)}\gamma_5 E_\beta^{(2)}$ it follows that only the first integral in (5.7) contibutes and we remain with

$$\mathrm{Tr}\Big(\gamma_5 g(D)\Big) = \int \mathrm{d}\Big(\mathrm{Tr}(\gamma_5 E_\alpha^{(5)})\Big)\; g(\alpha)\;. \tag{5.8}$$

Because from (5.5) we have $[\gamma_5, E_\alpha^{(5)}] = 0$ we can decompose $E_\alpha^{(5)}$ as $E_\alpha^{(5)} = E_\alpha^{(+)} + E_\alpha^{(-)}$ with $\gamma_5 E_\alpha^{(\pm)} = \pm E_\alpha^{(\pm)}$ so that (5.8) becomes

$$\mathrm{Tr}(\gamma_5 g(D)) = \int \mathrm{d}\Big(\mathrm{Tr}(E_\alpha^{(+)} - E_\alpha^{(-)})\Big)\; g(\alpha)\;. \tag{5.9}$$

Further, in (5.9) we can separate discrete and continuous parts of the spectum putting $E_\alpha^{(\pm)} = E_\alpha^{\mathrm{d}(\pm)} + E_\alpha^{\mathrm{c}(\pm)}$. With $\int \mathrm{d}\Big(\mathrm{Tr}\, E_\alpha^{\mathrm{d}(\pm)}\Big)\; g(\alpha) = \sum_j g(\alpha_j) d_j^{(\pm)}$ and $I(\alpha_j) = d_j^{(+)} - d_j^{(-)}$ this gives

$$\mathrm{Tr}(\gamma_5 g(D)) = \sum_j g(\alpha_j) I(\alpha_j) + \int \mathrm{d}\Big(\mathrm{Tr}(E_\alpha^{\mathrm{c}(+)} - E_\alpha^{\mathrm{c}(-)})\Big)\; g(\alpha) \tag{5.10}$$

which generalizes (3.10).

An important observation is now that for the large class of spectra with only discrete points on the real axis still only the discrete spectrum contributes to (5.10). This follows because $E_\alpha^{\mathrm{c}(\pm)}$ then must be constant outside the respective points and, on the other hand, by definition is continuous.

It should be noted that the indicated class includes the cases where the spectrum is restricted to curves which cross the real axis. This occurs, for example, with the circle in the simple GW case. Generally this can be achieved by imposing appropriate constraints as discussed in Section 3.4.

5.4 $\mathrm{Tr}\,\gamma_5$ in Hilbert space

The chiral Ward identities are particular decompositions of $\mathrm{Tr}\,\gamma_5$ in the global case and of $\mathrm{Tr}\,\gamma_5 \hat{e}(n)$ in the local one. That these traces are zero makes the respective relations identities. We have to check if this is also guaranteed in the infinite-volume limit.

In the local case this clearly holds since $\gamma_5 \hat{e}(n)$ is restricted to a subspace of finite dimension. In the global case we have, firstly, to be consistent with the fact that the infinite-lattice result is to be considered as limit of the results on finite lattices with increasing size, which for $\mathrm{Tr}\,\gamma_5$ means that we get a sequence with all members zero and therefore also zero in the limit. Secondly, consistency is to be required with the fact that summing the local result over n we get the global one which with respect to $\mathrm{Tr}\,\gamma_5$ again amounts to a sequence with all members and thus also the limit being zero. Thirdly, with $\{\gamma_\mu,\gamma_5\}=0$ and $\gamma_\mu^2=1$ one in the usual way gets $\mathrm{Tr}\,\gamma_5=-\mathrm{Tr}\,\gamma_5$ and can hardly escape to conclude that $\mathrm{Tr}\,\gamma_5=0$. Thus not to run into severe contradictions we must still have the relation $\mathrm{Tr}\,\gamma_5=0$ in the limit.

We now consider what this in more detail means in Hilbert space. For this purpose we first remember that $\mathrm{Tr}\,\gamma_5$ is actually a shorthand for $\mathrm{Tr}(\gamma_5\otimes\mathbb{1}_s)$, or introducing $\Gamma_5=\gamma_5\otimes\mathbb{1}_s$, of $\mathrm{Tr}\;\Gamma_5=0$. We also note that the projection involved in the local case is of form $\tilde{P}_j=\mathbb{1}_\gamma\otimes\tilde{p}_j$. In this form we are free to choose, more generally, any set of orthogonal projections $\tilde{p}_j$ which project onto a subspaces of finite dimension. With $\sum_{j=1}^{\infty}\tilde{p}_j=\mathbb{1}_s$ we then have $\sum_{j=1}^{\infty}\tilde{P}_j=\mathbb{1}$. Now clearly $\mathrm{Tr}(\Gamma_5\sum_{j=1}^{N}\tilde{P}_j)=0$ holds for finite N. Thus we can define $\mathrm{Tr}\,\Gamma_5$ by the limit of the sequence of trace class operators $\Gamma_5\sum_{j=1}^{N}\tilde{P}_j$ as

$$\mathrm{Tr}(\Gamma_5)=\lim_{N\to\infty}\mathrm{Tr}(\Gamma_5\sum_{j=1}^{N}\tilde{P}_j) \tag{5.11}$$

or more simply, performing the traces first, by

$$\mathrm{Tr}(\Gamma_5)=\sum_{j=1}^{\infty}\mathrm{Tr}(\Gamma_5\tilde{P}_j)\;. \tag{5.12}$$

Both of these regularizations give $\mathrm{Tr}\,\Gamma_5=0$.

The relation $\mathrm{Tr}\,\Gamma_5=0$ actually expresses the fact that there is no asymmetry of the chiral subspaces. To make this explicit we decompose $\tilde{P}_j$ as $\tilde{P}_j=\tilde{P}_j^{(+)}-\tilde{P}_j^{(-)}$ where $\tilde{P}_j^{(\pm)}=\frac{1}{2}(1\pm\gamma_5)\otimes\tilde{p}_j$. We then have $\Gamma_5\tilde{P}_j=\tilde{P}_j^{(+)}-\tilde{P}_j^{(-)}$ and

$$\mathrm{Tr}\Big(\Gamma_5\tilde{P}_j\Big)=\mathrm{Tr}\,\tilde{P}_j^{(+)}-\mathrm{Tr}\,\tilde{P}_j^{(-)}=0 \tag{5.13}$$

which means that the subspaces onto which $\tilde{P}_j^{(+)}$ and $\tilde{P}_j^{(-)}$ project have the same dimension. Since the total space is made up of such pairs of chiral subspaces it is seen that, with $\tilde{P}^{(\pm)}=\sum_{j=1}^{\infty}\tilde{P}_j^{(\pm)}$, the operators $\tilde{P}^{(+)}$ and $\tilde{P}^{(-)}$ project onto spaces which have the same cardinality. Summing up (5.13) we get again $\mathrm{Tr}\,\Gamma_5=0$.

Mathematically another definition of the equality of cardinalities is that by the existence of an appropriate bijective mapping [24]. This is provided here by $\Gamma_4=\gamma_4\otimes\mathbb{1}_s$ with which one gets $\Gamma_4\tilde{P}_j^{(\pm)}\Gamma_4=\tilde{P}_j^{(\mp)}$ as well as $\Gamma_4\tilde{P}^{(\pm)}\Gamma_4=\tilde{P}^{(\mp)}$.

5.5 Limit $a\to 0$ of gauge fields

The operators $\mathcal{U}_\mu$ in Hilbert space which involve the gauge fields in n-representation are given by

$$(\mathcal{U}_\mu)_{n'n}=U_{\mu n}\delta^4_{n',n+\hat{\mu}}\;. \tag{5.14}$$

With them we get four unitary plaquette operators defined by $\mathcal{P}^{(1)}_{\mu\nu} = \mathcal{U}^\dagger_\mu \mathcal{U}^\dagger_\nu \mathcal{U}_\mu \mathcal{U}_\nu$ and

$$\mathcal{P}^{(2)}_{\mu\nu} = \mathcal{U}_\nu \mathcal{P}^{(1)}_{\mu\nu} \mathcal{U}^\dagger_\nu\,,\ \mathcal{P}^{(3)}_{\mu\nu} = \mathcal{U}_\mu \mathcal{P}^{(2)}_{\mu\nu} \mathcal{U}^\dagger_\mu\,,\ \mathcal{P}^{(4)}_{\mu\nu} = \mathcal{U}^\dagger_\nu \mathcal{P}^{(3)}_{\mu\nu} \mathcal{U}_\nu\,,\ \mathcal{P}^{(1)}_{\mu\nu} = \mathcal{U}^\dagger_\mu \mathcal{P}^{(4)}_{\mu\nu} \mathcal{U}_\mu \tag{5.15}$$

in terms of which we can introduce the field operators

$$\mathcal{F}^{(\alpha)}_{\mu\nu} = \mathbb{1} - \mathcal{P}^{(\alpha)\dagger}_{\mu\nu} \tag{5.16}$$

which have the advantageous property of being site diagonal

$$\left(\mathcal{F}^{(\alpha)}_{\mu\nu}\right)_{n'n} = -i\delta^4_{n'n} F^{(\alpha)}_{\mu\nu,n}\ . \tag{5.17}$$

The $F^{(\alpha)}_{\mu\nu,n}$ depend in an obvious way only on the $U_{\mu n}$. Further, using the unitary operators $\mathcal{T}_\mu$ which in n-representation are given by

$$(\mathcal{T}_\mu)_{n'n} = \delta^4_{n',n+\hat{\mu}} \tag{5.18}$$

one gets the site-diagonal combinations $(\mathcal{T}^\dagger_\mu \mathcal{U}_\mu)_{n'n} = \delta^4_{n'n} U_{\mu n}$ and

$$(\mathcal{T}^\dagger_\lambda \mathcal{F}^{(\alpha)}_{\mu\nu} \mathcal{T}_\lambda)_{n'n} = -i\delta^4_{n'n} F^{(\alpha)}_{\mu\nu,n+\hat{\lambda}} \quad , \quad (\mathcal{T}_\lambda \mathcal{F}^{(\alpha)}_{\mu\nu} \mathcal{T}^\dagger_\lambda)_{n'n} = -i\delta^4_{n'n} F^{(\alpha)}_{\mu\nu,n-\hat{\lambda}} \tag{5.19}$$

which will be also needed in the following.

With $U_{\mu n} = \exp(iB_{\mu n})$ we now list the minimal conditions on the $B_{\mu n}$ which will be seen to be necessary in the limit. Firstly we require

$$B_{\mu n}/a \to A_\mu \quad \text{and} \quad an \to x \quad \text{for} \quad a \to 0 \tag{5.20}$$

which, as already stressed in Section 5.1, implies that at the same time $n \to \infty$. By (5.20) the values A_μ and x are related which constitutes a pointwise definition of the matrix function $A_\mu(x)$. Secondly

$$(B_{\mu,n+\hat{\nu}} - B_{\mu n})/a^2 \to b_{\mu\nu} \quad \text{and} \quad an \to x \quad \text{for} \quad a \to 0 \tag{5.21}$$

is to be required for $\nu \neq \mu$. The values $b_{\mu\nu}$ and x in (5.21) then provide a pointwise definition of the matrix function $\partial A_\mu(x)/\partial x_\nu$ with $\nu \neq \mu$. In this way in addition $A_\mu(x)$ gets continuous in ν-direction. Thirdly we must have

$$(B_{\mu,n} - B_{\mu,n\mp\hat{\mu}})/a \to 0 \quad \text{and} \quad an \to x \quad \text{for} \quad a \to 0\ , \tag{5.22}$$

which provides continuity of $A_\mu(x)$ also in μ-direction. Fourthly we need

$$(B_{\mu,n} - B_{\mu,n-\hat{\mu}} - B_{\mu,n\mp\hat{\lambda}} + B_{\mu,n-\hat{\mu}-\mp\hat{\lambda}})/a^2 \to 0 \quad \text{and} \quad an \to x \quad \text{for} \quad a \to 0 \tag{5.23}$$

for $\nu \neq \mu$ giving continuity of $\partial A_\mu(x)/\partial x_\nu$ in all directions.

Using the explicit form of $F^{(\alpha)}_{\mu\nu,n}$ and the Baker-Campbell-Hausdorff formula one finds that with these conditions one gets independently of α

$$F^{(\alpha)}_{\mu\nu,n}/a^2 \to F_{\mu\nu} \quad \text{and} \quad an \to x \quad \text{for} \quad a \to 0 \tag{5.24}$$

where the values $F_{\mu\nu}$ and x provide the pointwise definition of the function $F_{\mu\nu}(x)$. On the other hand, $F_{\mu\nu}(x)$ then is in the usual way given by the functions $A_\mu(x)$ and $\partial A_\mu(x)/\partial x_\nu$ which follow here from their pointwise definitions. Thus $F_{\mu\nu}(x)$ is also continuous. In addition these conditions guarantee that one has also

$$F^{(\alpha)}_{\mu\nu,n\pm\hat{\lambda}}/a^2 \to F_{\mu\nu} \quad \text{and} \quad an \to x \quad \text{for} \quad a \to 0 \tag{5.25}$$

for the translated fields of (5.19), which we shall need too. Conversely, in order that (5.24) and (5.25) hold all of the indicated conditions are necessary in full.

5.6 Gauge-field norms

In the following we shall need a appropriate behaviors of gauge-field norms. We therefore in addition require the quantities A_μ and $b_{\mu\nu}$ in (5.20) and (5.21) to be bounded, which implies boundedness of $A_\mu(x)$ and $F_{\mu\nu}(x)$.

With (5.17) the norm squared of the operators $\mathcal{F}^{(\alpha)}_{\mu\nu}$ gets in n-representation

$$||\mathcal{F}^{(\alpha)}_{\mu\nu}||^2 = \sup_\phi \sum_n \phi_n^\dagger F^{(\alpha)\dagger}_{\mu\nu,n} F^{(\alpha)}_{\mu\nu,n} \phi_n \tag{5.26}$$

where $\sum_n \phi_n^\dagger \phi_n = 1$. In the limit $a \to 0$ with (5.24) we obtain in x-representation

$$||\mathcal{F}^{(\alpha)}_{\mu\nu}||^2 \to a^4 \sup_\phi \int \mathrm{d}^4x\, \phi^\dagger(x) F_{\mu\nu}(x)^2 \phi(x) \tag{5.27}$$

with $\int \mathrm{d}^4x\, \phi(x)^\dagger \phi(x) = 1$. Thus, since $F_{\mu\nu}(x)$ is bounded, the norm vanishes in the limit.

Further, for the norm squared of the site diagonal combinations $\mathcal{T}_\mu^\dagger \mathcal{U}_\mu - 1\!\!1$ we have

$$||\mathcal{T}_\mu^\dagger \mathcal{U}_\mu - 1\!\!1||^2 = \sup_\phi \sum_n \phi_n^\dagger (2 - U_{\mu n} - U_{\mu n}^\dagger) \phi_n \ . \tag{5.28}$$

In the limit with (5.20) we now obtain in x-representation

$$||\mathcal{T}_\mu^\dagger \mathcal{U}_\mu - 1\!\!1||^2 \to a^2 \sup \int \mathrm{d}^4x\, \phi^\dagger(x) A_\mu(x)^2 \phi(x) \tag{5.29}$$

which with the boundedness of $A_\mu(x)$ also vanishes for $a \to 0$.

6. Limit of index relation

6.1 Form of anomaly term

To calculate the limit $a \to 0$ of the anomaly term in the index relation we have to choose an explicit lattice form of D. In principle there is a large class of appropriate ones with the same limit out of which we could take one. In practice, however, the only well-established explicit form available is the overlap Dirac operator [4]

$$D = \rho\Big(1 + \gamma_5 \epsilon(H)\Big) \tag{6.1}$$

where $\epsilon(H)$ is the sign function and H the hermitean Wilson-Dirac operator. The operator (6.1) is normal and γ_5-hermitean. It satisfies the GW relation (4.4) and in addition to zero allows for the real eigenvalue 2ρ. We also note that with this operator by (5.10) only the discrete spectrum contributes to the index relation. Thus it is a suitable choice.

Since (6.1) satisfies (4.4) the index term simplifies to the form (4.6) which by inserting (6.1) becomes

$$\frac{1}{2}\mathrm{Tr}(\epsilon(H)) \; . \tag{6.2}$$

Obviously ρ has dropped out in (6.2) so that the particular value of $\rho \neq 0$ in (6.1) is irrelevant in the present context.

For H in (6.1) we use the hermitean Wilson-Dirac operator of form

$$H = \gamma_5\Big(\sum_\mu (\gamma_\mu D_\mu + W_\mu) - 1\Big) \tag{6.3}$$

in which the Wilson parameter is $r = 1$ and where for m the favorable value [18] $m = -1$ has been chosen. In (6.3) we have

$$D_\mu = \frac{1}{2}(\mathcal{U}_\mu^\dagger - \mathcal{U}_\mu) \quad , \quad W_\mu = 1\!\!1 - \frac{1}{2}(\mathcal{U}_\mu^\dagger + \mathcal{U}_\mu) \tag{6.4}$$

the $\mathcal{U}_\mu$ being given by (5.14).

Since H is a selfadjoint operator in Hilbert space we can express the index term (6.2) as

$$\frac{1}{2}\mathrm{Tr}(\epsilon(H)) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}\, s \; \mathrm{Tr}\frac{H}{H^2 + s^2} \; . \tag{6.5}$$

Decomposing H^2 into parts with and without γ-matrices gives $H^2 = L - V$ with

$$L = 1\!\!1 + \sum_{\mu \neq \nu} W_\mu W_\nu \; , \tag{6.6}$$

$$V = \sum_{\mu \neq \nu} \Big(\frac{1}{2}\gamma_\mu\gamma_\nu [D_\mu, D_\nu] + \gamma_\mu [D_\mu, W_\nu]\Big) \; . \tag{6.7}$$

Because at least four γ-matrices are needed to contribute to the trace we can write

$$\mathrm{Tr}\frac{H}{H^2+s^2} = \mathrm{Tr}\Big(HGVGVG(\mathbb{1}+V\frac{1}{L+s^2-V})\Big) \tag{6.8}$$

where $G=(L+s^2)^{-1}$.

6.2 Limit of anomaly term

The task now is to perform the $a\to 0$ limit of (6.5) with (6.8). To get rid of the last term in (6.8) we show that we have the norm convergence $\mathbb{1}+V(L+s^2-V)^{-1}\Rightarrow\mathbb{1}$ for $a\to 0$, which requires that that $||V(L+s^2-V)^{-1}||\to 0$. According to $||V(L+s^2-V)^{-1}||\le ||V||\,||(L+s^2-V)^{-1}||$ this follows if $||V||\to 0$ and if $(L+s^2-V)^{-1}$ is bounded. To derive this boundedness we first split off from L a positive part, $L=1+L_++L_F$, where with $\nabla_\mu=\mathbb{1}-\mathcal{U}_\mu$

$$L_+ = \frac{1}{8}\sum_{\mu\neq\nu}\Big(\nabla_\mu\nabla_\nu(\nabla_\mu\nabla_\nu)^\dagger+\nabla_\mu^\dagger\nabla_\nu(\nabla_\mu^\dagger\nabla_\nu)^\dagger\Big)\,, \tag{6.9}$$

$$L_F = -\frac{1}{8}\sum_{\mu\neq\nu}\Big(\mathcal{U}_\nu(\mathcal{F}^{(1)}_{\mu\nu}+\mathcal{F}^{(4)\dagger}_{\mu\nu})+(\mathcal{U}_\nu(\mathcal{F}^{(1)}_{\mu\nu}+\mathcal{F}^{(4)\dagger}_{\mu\nu}))^\dagger\Big)\,. \tag{6.10}$$

We then note that due to the selfadjointness of the operators we obtain the ordering relation $(L+s^2-V)\ge 1+L_F-V\ge 1-||L_F||-||V||$. It implies $(L+s^2-V)^{-1}\le 1-||L_F||-||V||$ from which one gets $||(L+s^2-V)^{-1}||\le 1-||L_F||-||V||$. Thus all we need is that $||L_F||\to 0$ and $||V||\to 0$. To deal with V we note that

$$\begin{aligned}
4[D_\mu,D_\nu] &= \mathcal{U}_\mu\mathcal{U}_\nu\mathcal{F}^{(1)}_{\mu\nu}+\mathcal{U}_\mu^\dagger\mathcal{U}_\nu^\dagger\mathcal{F}^{(3)}_{\mu\nu}+\mathcal{U}_\nu\mathcal{U}_\mu^\dagger\mathcal{F}^{(4)}_{\mu\nu}+\mathcal{U}_\nu^\dagger\mathcal{U}_\mu\mathcal{F}^{(2)}_{\mu\nu}\,,\\
4[D_\mu,W_\nu] &= \mathcal{U}_\mu\mathcal{U}_\nu\mathcal{F}^{(1)}_{\mu\nu}-\mathcal{U}_\mu^\dagger\mathcal{U}_\nu^\dagger\mathcal{F}^{(3)}_{\mu\nu}+\mathcal{U}_\nu\mathcal{U}_\mu^\dagger\mathcal{F}^{(4)}_{\mu\nu}-\mathcal{U}_\nu^\dagger\mathcal{U}_\mu\mathcal{F}^{(2)}_{\mu\nu}\,.
\end{aligned} \tag{6.11}$$

From (6.10) and (6.11) it becomes obvious that $||L_F||\to 0$ and $||V||\to 0$ follow from $||\mathcal{F}^{(\alpha)}_{\mu\nu}||\to 0$, which we have by (5.27). In (6.8) we thus remain with $\mathrm{Tr}\Big(HGVGVG\Big)$ and after evaluation of the γ-traces get

$$\begin{aligned}
\mathrm{Tr}\frac{H}{H^2+s^2} \to \sum_{\mu\nu\lambda\tau\sigma}\epsilon_{\mu\nu\lambda\tau}\mathrm{Tr}\Big(&G(W_\sigma-1/4)G[D_\mu,D_\nu]G[D_\lambda,D_\tau]\\
&+2GD_\mu G[D_\nu,W_\sigma]G[D_\lambda,D_\tau]+2G[D_\nu,W_\sigma]GD_\mu G[D_\lambda,D_\tau]\Big)\,.
\end{aligned} \tag{6.12}$$

To evaluate the commutators in (6.12) we use that all $F^{(\alpha)}_{\mu\nu,n}$ by (5.24) lead to the same limit so that in (6.11) we can replace all $\mathcal{F}^{(\alpha)}_{\mu\nu}$ by one of them. Further the $\mathcal{U}_\mu$ factors there can be replaced by the $\mathcal{T}_\mu$ since $\mathcal{U}_\mu\Rightarrow\mathcal{T}_\mu$ according to (5.29). Thus instead of (6.11) we now have

$$4[D_\mu,D_\nu]\Rightarrow(\mathcal{T}_\mu+\mathcal{T}_\mu^\dagger)(\mathcal{T}_\nu+\mathcal{T}_\nu^\dagger)\mathcal{F}^{(1)}_{\mu\nu}\,,\quad 4[D_\mu,W_\nu]\Rightarrow(\mathcal{T}_\mu+\mathcal{T}_\mu^\dagger)(\mathcal{T}_\nu-\mathcal{T}_\nu^\dagger)\mathcal{F}^{(1)}_{\mu\nu}\,. \tag{6.13}$$

In (6.12) we further can replace G by $G^{(0)} = (L^{(0)} + s^2)^{-1}$ where $L^{(0)} = \mathbb{1} + \sum_{\mu\neq\nu} W_\mu^{(0)} W_\nu^{(0)}$ with $W_\mu^{(0)} = \mathbb{1} - \frac{1}{2}(\mathcal{T}_\mu^\dagger + \mathcal{T}_\mu)$ since $G \Rightarrow G^{(0)}$, as follows from $G = G^{(0)}(\mathbb{1} - (L - L^{(0)})G)$ because of $||(L - L^{(0)})|| \to 0$ and boundedness of G. This boundedness is obtained from the ordering relation $(L+s^2) \geq 1 + L_F \geq 1 - ||L_F||$ which implies $||(L+s^2)^{-1}|| \leq 1 - ||L_F||$. That $||(L - L^{(0)})|| \to 0$ follows from (5.29) noting that $||\mathcal{U}_\mu\mathcal{U}_\nu - \mathcal{T}_\mu\mathcal{T}_\nu|| \leq ||\mathcal{T}_\mu^\dagger\mathcal{U}_\mu - \mathbb{1}|| + ||\mathcal{T}_\nu^\dagger\mathcal{U}_\nu - \mathbb{1}|| + ||\mathcal{T}_\mu^\dagger\mathcal{U}_\mu - \mathbb{1}||\,||\mathcal{T}_\nu^\dagger\mathcal{U}_\nu - \mathbb{1}||$. Because of $||(\mathcal{U}_\mu^\dagger \mp \mathcal{U}_\mu) - (\mathcal{T}_\mu^\dagger \mp \mathcal{T}_\mu)|| \leq 2||\mathcal{T}_\mu^\dagger\mathcal{U}_\mu - \mathbb{1}||$ and (5.29) we can also replace $GD_\mu G$ by $G^{(0)2}(\mathcal{T}_\mu^\dagger - T_\mu)/2$ and $G(1-W_\sigma)G$ by $G^{(0)2}(\mathcal{T}_\sigma^\dagger + T_\sigma)/2$ in (6.12). Further, because of (5.19) with (5.25), we can interchange there $\mathcal{F}_{\mu\nu}^{(1)}$ with the $\mathcal{T}_\lambda$ and the $\mathcal{T}_\lambda^\dagger$, and also with $G^{(0)}$ which is a function of such operators. In addition we make use of the fact that $\mathcal{F}_{\mu\nu}^{(1)}$ and $-\mathcal{F}_{\nu\mu}^{(1)}$ have the same limit.

With the mentioned replacements and interchanges (6.12) becomes

$$G^{(0)3} \sum_{\mu\nu\lambda\tau} \epsilon_{\mu\nu\lambda\tau} \mathrm{Tr}\biggl(\sum_\sigma \Bigl(\frac{(\mathcal{T}_\sigma^\dagger - \mathcal{T}_\sigma)^2}{2(\mathcal{T}_\sigma^\dagger + \mathcal{T}_\sigma)} + \frac{3}{4} - \frac{\mathcal{T}_\sigma^\dagger + \mathcal{T}_\sigma}{2} \Bigr) \prod_\rho \frac{\mathcal{T}_\rho^\dagger + \mathcal{T}_\rho}{2} \mathcal{F}_{\mu\nu}^{(1)} \mathcal{F}_{\lambda\tau}^{(1)} \biggr) . \tag{6.14}$$

Remembering $L = 1 + L_+ + L_F$ with (6.9) and (6.10) we see that $L_0 \geq \mathbb{1}$. Thus inserting (6.14) into (6.5) we can perform the integral $\int \mathrm{d}s\, G^{(0)3}$ and arrive at

$$\frac{1}{2} \mathrm{Tr}\, \epsilon(H) \to \sum_{\mu\nu\lambda\tau} \epsilon_{\mu\nu\lambda\tau} \mathrm{Tr}\Bigl(\mathcal{C} \mathcal{F}_{\mu\nu}^{(1)} \mathcal{F}_{\lambda\tau}^{(1)} \Bigr) \tag{6.15}$$

where

$$\mathcal{C} = \frac{3}{16} \sum_\sigma \left(\frac{(\mathcal{T}_\sigma^\dagger - \mathcal{T}_\sigma)^2}{2(\mathcal{T}_\sigma^\dagger + \mathcal{T}_\sigma)} + \frac{3}{4} - \frac{\mathcal{T}_\sigma^\dagger + \mathcal{T}_\sigma}{2} \right) \prod_\rho \frac{\mathcal{T}_\rho^\dagger + \mathcal{T}_\rho}{2} \frac{1}{(\sqrt{L^{(0)}}\,)^5} . \tag{6.16}$$

With (5.17) we then have $\mathrm{Tr}(\mathcal{C}\mathcal{F}_{\mu\nu}^{(1)}\mathcal{F}_{\lambda\tau}^{(1)}) = -\sum_n \mathcal{C}_{nn} \mathrm{tr}(F_{\mu\nu,n}^{(1)} F_{\lambda\tau,n}^{(1)})$. Since $\mathcal{C}$ depends on the $\mathcal{T}_\mu$ only, $\mathcal{C}_{nn}$ is a number independent of n. It is obviously given by

$$\mathcal{C}_{nn} = -\frac{3}{16} \prod_\rho \Bigl(\frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\, k_\rho \cos k_\rho \Bigr) \frac{1 + \sum_\sigma (\cos k_\sigma - 1 + \sin^2 k_\sigma / \cos k_\sigma)}{\Bigl(\sqrt{1 + \sum_{\mu\neq\nu} (1 - \cos k_\mu)(1 - \cos k_\nu)}\, \Bigr)^5} . \tag{6.17}$$

The integral in (6.17) can be evaluated in the way introduced in [25] and also used in [7, 8], which gives $\mathcal{C}_{nn} = -(32\pi^2)^{-1}$. Inserting (5.24) we now have for $a \to 0$ in x-representation

$$\frac{1}{2} \mathrm{Tr}\, \epsilon(H) \to \frac{1}{32\pi^2} \int \mathrm{d}^4 x \sum_{\mu\nu\lambda\tau} \epsilon_{\mu\nu\lambda\tau}\ \mathrm{tr}\Bigl(F_{\mu\nu}(x) F_{\lambda\tau}(x) \Bigr) . \tag{6.18}$$

6.3 Index theorem

With the Dirac operator (6.1) by (5.10) only the discrete spectrum contributes to the identity (2.12). The index term then by (3.12) is $I(0)$ as defined in (3.11). The anomaly term for the operator (6.1) simplifies to (4.6) and gets the form (6.2) which has the limit (6.18). We thus obtain

$$\mathrm{Tr}\,\gamma_5 = I(0) + \frac{1}{32\pi^2}\int \mathrm{d}^4x \sum_{\mu\nu\lambda\tau} \epsilon_{\mu\nu\lambda\tau}\ \mathrm{tr}\Big(F_{\mu\nu}(x)F_{\lambda\tau}(x)\Big) = 0 \tag{6.19}$$

which is the index theorem as it follows in the framework of nonperturbative quantized theory on the basis of the chiral Ward identity.

Since $I(0)$ takes integer values only this must also hold for the gauge-field term in (6.19). With respect to the topological side of this we remember that the minimal conditions on the gauge fields needed for the limit of the anomaly term have precisely lead to continuity of $F_{\mu\nu}(x)$. Because continuity is a prerequisite for homotopy classes and thus for topological invariants this hints at the underlying mechanism.

While so far we have needed only boundedness of $F_{\mu\nu}(x)$, to get a finite result in (6.18) it should decrease sufficiently fast at infinity. However, admitting that the index also may get infinite this is not necessary.

7. Comparison with Atiyah-Singer theorem

7.1 Form of theorem

The Atiyah-Singer theorem for elliptic differential operators on compact manifolds without boundaries relates the analytical index of such operators to topological invariants. The relevant papers[4] are [16, 17]. Ref. [16] gives the proof in K-theory (i.e. entirely within algebraic topology) and Ref. [17] mainly translates the results to cohomology. The special case of the Dirac operator, to be considered here, is treated in Section 5 of [17].

In the case of interest here the theorem says that the index of the Weyl operator $D^{(+)}$ associated to the Dirac operator D_{A} equals the Pontryagin index, which we may write as

$$\text{index } D^{(+)} = -\frac{1}{32\pi^2}\int \mathrm{d}^4x \sum_{\mu\nu\lambda\tau} \epsilon_{\mu\nu\lambda\tau}\ \mathrm{tr}\Big(F_{\mu\nu}(x)F_{\lambda\tau}(x)\Big) \tag{7.1}$$

where index $D^{(+)} =$ dim ker $D^{(+)} -$ dim ker $D^{(+)\dagger}$. The operator $D^{(+)}$ maps between two spinor spaces, i.e. from $E^{(+)}$ to $E^{(-)}$ say, and its adjoint $D^{(+)\dagger} \equiv D^{(-)}$ back [17].

[4]Further papers contain different types of proofs or present various generalizations. Other ones, also occasionally referred to, actually consider manifolds with boundaries.

7.2 Analytical relations

For our comparison we need to know what in detail holds for the analytical index. To get the connection to zero modes one has to note that $D^{(-)}D^{(+)}$ maps within $E^{(+)}$ and that $D^{(+)}D^{(-)}$ maps within $E^{(-)}$. Both of these operators are selfadjoint and nonnegative. Since one is on a compact manifold their spectra are discrete and the degeneracies for nonzero eigenvalues are finite. Except for zero modes one has the same spectra for both operators. This follows since, firstly, with the eigenequation $D^{(-)}D^{(+)}\varphi_j = \kappa_j\varphi_j$ in $E^{(+)}$ multiplying by $D^{(+)}$ one gets the eigenequation $D^{(+)}D^{(-)}(D^{(+)}\varphi_j) = \kappa_j(D^{(+)}\varphi_j)$ in $E^{(-)}$. Secondly, in case of degeneracy at an eigenvalue κ_j considering $\langle D^{(+)}\varphi_{jr'}|D^{(+)}\varphi_{jr}\rangle = \langle\varphi_{jr'}|D^{(-)}D^{(+)}\varphi_{jr}\rangle = \kappa_j\langle\varphi_{jr'}|\varphi_{jr}\rangle$ one sees that the eigenspaces must have the same dimension exept for $\kappa_j = 0$.

For $\kappa_j > 0$ we thus have $\dim E_j^{(+)} = \dim E_j^{(-)}$ or in terms of eigenprojections

$$\mathrm{Tr}\Big(P_j^{(+)} - P_j^{(-)}\Big) = 0 \quad \text{for} \quad \kappa_j > 0\ . \tag{7.2}$$

To see what happens for $\kappa_j = 0$ one has to note that $\ker D^{(+)} = \ker D^{(-)}D^{(+)}$ and $\ker D^{(-)} = \ker D^{(+)}D^{(-)}$, which from left to right is obvious and in the opposite direction follows from $\langle\varphi|D^{(-)}D^{(+)}\varphi\rangle = \langle D^{(+)}\varphi|D^{(+)}\varphi\rangle$. We therefore for $\kappa_j = 0$ get $\dim E_j^{(\pm)} = \dim\ker D^{(\pm)}$ and in terms of eigenprojections

$$\mathrm{index}\, D^{(+)} = \mathrm{Tr}\Big(P_j^{(+)} - P_j^{(-)}\Big) \quad \text{for} \quad \kappa_j = 0\ . \tag{7.3}$$

We thus make the remarkable observation that for a nonvanishing index of $D^{(+)}$ the dimensions of the eigenspaces $E_j^{(+)}$ and $E_j^{(-)}$ for $\kappa_j = 0$ are different while for $\kappa_j > 0$ they are always equal. This means that for a nonvanishing index the spaces $E^{(+)}$ and $E^{(-)}$ have different cardinalities.

Defining $P^{(\pm)} = \sum_j P_j^{(\pm)}$ the projections $P^{(+)}$ and $P^{(-)}$ are the ones projecting onto the spaces $E^{(+)}$ and $E^{(-)}$, respectively. Combining (7.2) and (7.3) one formally gets $\mathrm{index}\, D^{(+)} = \mathrm{Tr}(P^{(+)} - P^{(-)})$. This can be defined as the limit of a sequence of trace-class operators $\sum_{j=1}^N (P_j^{(+)} - P_j^{(-)})$ by

$$\mathrm{index}\, D^{(+)} = \lim_{N\to\infty} \mathrm{Tr}\sum_{j=1}^{N}\Big(P_j^{(+)} - P_j^{(-)}\Big) \tag{7.4}$$

or more simply, performing the traces first, by

$$\mathrm{index}\, D^{(+)} = \sum_{j=1}^{\infty} \mathrm{Tr}\Big(P_j^{(+)} - P_j^{(-)}\Big)\ . \tag{7.5}$$

Obviously the regularizations in (7.4) and (7.5) are analougous to those in (5.11) and (5.12), however, the space structure here is fundamentally different.

7.3 Dirac operator

The Dirac operator D_{A} is defined as the composition of the two maps $D^{(+)}$ and $D^{(-)}$ [17]. Thus in the combined space $E = E^{(+)} \bigoplus E^{(-)}$ introducing

$$\hat{D}^{(\pm)} = \begin{cases} D^{(\pm)} & \text{for mapping from } E^{(\pm)} \text{ to } E^{(\mp)} \\ 0 & \text{otherwise} \end{cases} \quad . \tag{7.6}$$

we have $D_{\mathrm{A}} = \hat{D}^{(+)} + \hat{D}^{(-)}$. Obviously D_{A} is selfadjoint. From (7.6) one obtains $D_{\mathrm{A}}^2 = \hat{D}^{(+)}\hat{D}^{(-)} + \hat{D}^{(-)}\hat{D}^{(+)}$. This connects to the eigenequations considered for the operators $D^{(+)}D^{(-)}$ and $D^{(-)}D^{(+)}$ before and leads to the spectral representation

$$D_{\mathrm{A}} = \sum_j \lambda_j (P_j^{(+)} + P_j^{(-)}) \quad \text{with} \quad \lambda_j^2 = \kappa_j \; . \tag{7.7}$$

Introducing a function $f(D_{\mathrm{A}}^2)$, (7.2) generalizes to

$$\mathrm{Tr}\Big(f(D_{\mathrm{A}}^2)(P_j^{(+)} - P_j^{(-)})\Big) = 0 \quad \text{for} \quad \kappa_j > 0 \tag{7.8}$$

and, in addition requiring $f(0) = 1$, (7.3) to

$$\operatorname{index} D^{(+)} = \mathrm{Tr}\Big(f(D_{\mathrm{A}}^2)(P_j^{(+)} - P_j^{(-)})\Big) \quad \text{for} \quad \kappa_j = 0 \; . \tag{7.9}$$

Because D_{A}^2 is nonnegative one can choose $f(D_{\mathrm{A}}^2)$ such that $f(D_{\mathrm{A}}^2)P^{(\pm)}$ get trace-class operators. This provides a convenient regularization with which one can readily combine (7.8) and (7.9) to

$$\operatorname{index} D^{(+)} = \mathrm{Tr}\Big(f(D_{\mathrm{A}}^2)(P^{(+)} - P^{(-)})\Big) \; . \tag{7.10}$$

This regularization is, for example, used with $f(D_{\mathrm{A}}^2) = \exp(-tD_{\mathrm{A}}^2)$ in an alternative proof of the Atiyah-Singer theorem based on the heat equation [26]. While technically the regularization in (7.10) has some advantage it clearly does not affect the structures of the spaces $E^{(+)}$ and $E^{(-)}$ in any way.

We note that D_{A} given by (7.6) anticommutes with $P^{(+)} - P^{(-)}$,

$$\{(P^{(+)} - P^{(-)})\,, D_{\mathrm{A}}\} = 0 \; . \tag{7.11}$$

We further find that the transformation (2.1) with

$$\Gamma = \bar{\Gamma} = P^{(+)} - P^{(-)} \tag{7.12}$$

leaves the classical action $\bar{\psi} D_{\mathrm{A}} \psi$ invariant. Obviously this is the global chiral transformation in the Atiyah-Singer case.

Though the Atiyah-Singer framework is a classical one we may compare the structures inserting its global chiral transformation with (7.12) into the identity (2.9) of the quantum case. We thus get

$$-\mathrm{Tr}\Big(P^{(+)} - P^{(-)}\Big) - \zeta\mathrm{Tr}\Big((D_{\mathrm{A}} - \zeta)^{-1}(P^{(+)} - P^{(-)})\Big) = 0 \tag{7.13}$$

which is made up of the measure contribution and of the index term of the action contribution while the anomaly term in the latter by (7.11) vanishes.

7.4 Relations with γ_5

In physical applications the connection of the relations in Sections 7.2 and 7.3 to ones with γ_5 is of interest. To make the respective details explicit we consider the projections $P_j^{(\pm)}$ onto the occurring eigenspaces. We are are free to express these operators in the forms $P_j^{(+)} = \begin{pmatrix} p_j^{(+)} & 0 \\ 0 & 0 \end{pmatrix}$ and $P_j^{(-)} = \begin{pmatrix} 0 & 0 \\ 0 & p_j^{(-)} \end{pmatrix}$. Introducing $\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ this can be written as

$$P_j^{(\pm)} = \frac{1}{2}(1 \pm \gamma_5) \bigotimes p_j^{(\pm)} . \tag{7.14}$$

For $\kappa_j > 0$ from (7.2) we have $\operatorname{Tr} P_j^{(+)} = \operatorname{Tr} P_j^{(-)}$ which is satisfied if $\operatorname{Tr} p_j^{(+)} = \operatorname{Tr} p_j^{(-)}$. Thus identifying $p_j^{(+)} = p_j^{(-)} = p_j$ in (7.14) we get a valid representation of $P_j^{(\pm)}$ and obtain

$$P_j^{(+)} - P_j^{(-)} = \gamma_5 \bigotimes p_j \quad \text{for} \quad \kappa_j > 0 . \tag{7.15}$$

For $\kappa_j = 0$ according to (7.3) we must allow for different values of $\operatorname{Tr} P_j^{(+)}$ and $\operatorname{Tr} P_j^{(-)}$. Then if $\operatorname{Tr} P_j^{(+)} > \operatorname{Tr} P_j^{(-)}$ we need $\operatorname{Tr} p_j^{(+)} > \operatorname{Tr} p_j^{(-)}$. To realize this we identify the space onto which $p_j^{(-)}$ projects with a subspace of that onto which $p_j^{(+)}$ projects so that $p_j^{(+)} p_j^{(-)} = p_j^{(-)} p_j^{(+)} = p_j^{(-)}$. The latter allows to decompose $p_j^{(+)}$ as $p_j^{(+)} = p_j^{(-)} + (p_j^{(+)} - p_j^{(-)})$ into orthogonal parts and inserting this decomposition into (7.14) gives $P_j^{(+)} - P_j^{(-)} = \gamma_5 \otimes p_j^{(-)} + \frac{1}{2}(1+\gamma_5) \otimes (p_j^{(+)} - p_j^{(-)})$. Proceeding analogously for $\operatorname{Tr} P_j^{(+)} < \operatorname{Tr} P_j^{(-)}$ we altogether have

$$P_j^{(+)} - P_j^{(-)} = \gamma_5 \bigotimes p_j^{(\mp)} + \frac{1}{2}(1 \pm \gamma_5) \bigotimes (p_j^{(+)} - p_j^{(-)}) \;\text{ for }\; \kappa_j = 0 \,,\; p_j^{(+)} \gtrless p_j^{(-)} . \tag{7.16}$$

Inserting now (7.15) and (7.16) into $P^{(+)} - P^{(-)} = \sum_j (P_j^{(+)} - P_j^{(-)})$ it becomes obvious that for a nonvanishing index of $D^{(+)}$ the extra term in (7.16) prevents $P^{(+)} - P^{(-)}$ from getting the form $\gamma_5 \otimes \mathbb{1}_s$, i.e. that we have

$$P^{(+)} - P^{(-)} = \gamma_5 \bigotimes \mathbb{1}_s \quad \text{iff} \quad \text{index}\, D^{(+)} = 0 . \tag{7.17}$$

Thus the naive dealing with γ_5 which effectively uses $\gamma_5 \otimes \mathbb{1}_s$ instead of $P^{(+)} - P^{(-)}$ turns out to be only valid if the index of $D^{(+)}$ vanishes.

7.5 Comparison of concepts

Clearly the settings to be compared are quite different. In the Atiyah-Singer case one considers a differential operator and the concept is that of classical fields, while in the lattice approach a subtle limit of a discrete operator realizes the quantum concept.

The basic structural difference is that in the Atiyah-Singer framework a nonvanishing index results from the fact that the dimensions of $E_j^{(+)}$ and $E_j^{(-)}$ for $\kappa_j = 0$ are different while those for all $\kappa_j > 0$ are the same. In other words, it stems from different cardinalities of the spaces $E^{(+)}$ and $E^{(-)}$, which in turn are solely caused by different dimensions of the chiral eigenspaces at $\kappa_j = 0$.

In contrast to this in nonperturbative quantized theory no such asymmetry of space exists. A nonvanishing index there results from a chirally noninvarinant part of the action which is conceptually necessary (to avoid doublers). The mechanism, reflected by the sum rule (3.14), then is that nonvanishing of the index necessarily implies the existence of a corresponding difference at other eigenvalues.

In the Atiyah-Singer case the space structure itself depends on D_{A}, i.e. on the particular gauge field. In fact, to determine the subspaces $E^{(+)}$ and $E^{(-)}$ which make E up, one has first to solve the respective eigenvalue problems. Obviously this has to be done for each gauge-field configuration and one has to allow for different structures in each case.

On the other hand, with the nonperturbative definition of the quantized theory the space structure is fixed and does not depend in any way on D and on the gauge field.

A further difference is that, while the formulation of the quantized theory given here is in $\mathbf{R^4}$, in the Atiyah-Singer case the fields live on a compact manifold[5].

8. Analysis of continuum approaches

8.1 General observations

In conventional continuum approaches to quantized theory the form $D = \sum_\mu \gamma_\mu D_\mu$ of the Dirac operator is used. For this form because of $\{\gamma_5, D\} = 0$ the anomaly term in the identity (2.12) vanishes. Further, since with hermitean γ-matrices the spectrum is on the imaginary axis according to (5.10) only the discrete point at zero contributes to the index term. Thus the identity (2.12) degenerates to

$$\mathrm{Tr}\,\gamma_5 = I(0) = 0\ . \tag{8.1}$$

For anti-hermitean γ-matrices[6], with the spectrum on the real axis and purely-imaginary-mode rules[3], again (8.1) is obtained.

Actually the result (8.1) is no surprise since proper nonperturbative definition of the quantized theory requires discretization also of the action. Then the choice $D = \sum_\mu \gamma_\mu D_\mu$ is not appropriate because it suffers from the doubling phenomenon and the vanishing of anomaly and index is exactly what in that case is to be expected.

[5]In [12] an extension to $\mathbf{R^4}$ has been proposed. It is, however, an open question wether the proof of [16] can be extended.

[6]Instead of the antihermitean operator D with hermitean γ-matrices γ_μ one may consider the hermitean operator $D^a = \sum_\mu \gamma^a_\mu D_\mu = iD$ with antihermitean γ-matrices $\gamma^a_\mu = i\gamma_\mu$.

One should also note that with a function $f(D^2)$ where $f(0)=1$ using (5.10) one gets $\mathrm{Tr}(\gamma_5 f(D^2)) = I(0)$. However, with (8.1) this gives

$$\mathrm{Tr}\Big(\gamma_5 f(D^2)\Big) = I(0) = 0 \tag{8.2}$$

so that the introduction of such a function is seen not to change the result.

In conventional continuum approaches to quantized theory instead of (7.12) of the Atiyah-Singer case one uses (2.10) with the global chiral transformation. Such replacement of $P^{(+)} - P^{(-)}$ by γ_5, however, according to (7.17) requires a vanishing index. Thus also from the Atiyah-Singer point of view (8.1) and (8.2) hold.

8.2 Perturbation theory

The question now is how in conventional continuum approaches nevertheless the chiral anomaly can arise. In perturbation theory at the level of the Ward identity (in the well known triangle diagram) one gets an ambiguity which, if fixed in a gauge-invariant way, produces the anomaly term [27, 28]. The point is that this fixing of the ambiguity constitutes a chirally noninvariant modification of the theory.

Thus there is no contradiction to the nonperturbative approach. One observes that while in continuum theory a modification occurs only at the level of the Ward identity and is put in by hand, in the nonperturbative theory based on the lattice it is already built in into the action and thus is included in the theory from the start.

8.3 Pauli-Villars approach

If in continuum perturbation theory the Pauli-Villars (PV) regularization is used, in the PV difference ambiguous contributions, being mass-independent, drop out so that the PV term gives the anomaly [27, 28]. This has suggested a nonperturbative interpretation of it based on an evalution of $-\lim_{m\to\infty} \mathrm{Tr}(\gamma_5 m(D+m)^{-1})$ as given in [11], which neglecting higher orders in the PV mass arrives at the desired result. If nonperturbatively correct this would be in contradiction to the fact that the anomaly term for $D=\sum_\mu \gamma_\mu D_\mu$ vanishes.

Using $\{\gamma_5, D\}=0$ one sees that $\mathrm{Tr}(\gamma_5 m(D+m)^{-1}) = \mathrm{Tr}(\gamma_5 m^2(m^2-D^2)^{-1})$ and further that this expression is independent of m. For $m\to 0$ it obviously gives the index $I(0)$. Actually it is just (8.2) with the choice $f(D^2) = m^2(m^2-D^2)^{-1}$ and therefore vanishes.

To check what has been done in [11] we decompose D^2 as $D^2 = \check{L} + \check{V}$ into parts with and without γ-matrices,

$$\check{L} = \sum_\mu D_\mu^2 \quad , \quad \check{V} = \frac{1}{2}\sum_{\mu\neq\nu} \gamma_\mu\gamma_\nu [D_\mu D_\nu] \ . \tag{8.3}$$

Because at least four γ-matrices are needed to contribute to the trace one can write

$$-\mathrm{Tr}\Big(\gamma_5 m(D+m)^{-1}\Big) = m^2\mathrm{Tr}\Big(\gamma_5\check{G}\check{V}\check{G}\check{V}\check{G}\Big) - m^2\mathrm{Tr}\Big(\gamma_5\check{G}\check{V}\check{G}\check{V}\check{G}\check{V}(D^2-m^2)^{-1}\Big) \quad (8.4)$$

where $\check{G} = (\check{L}-m^2)^{-1}$. In [11] only the term $m^2\mathrm{Tr}\Big(\gamma_5\check{G}\check{V}\check{G}\check{V}\check{G}\Big)$ of (8.4) is kept and evaluating it for $m^2 \to \infty$ gives the desired result.

The problem is, however, that neglecting the rest is not allowed because

$$m^2\mathrm{Tr}\Big(\gamma_5\check{G}\check{V}\check{G}\check{V}\check{G}\check{V}(D^2-m^2)^{-1}\Big) = m^2\mathrm{Tr}\Big(\gamma_5\check{G}\check{V}\check{G}\check{V}\check{G}\Big) \quad (8.5)$$

holds, which reflects the fact that the correct nonperturbative result is zero. Thus neglegting the respective term is actually a modification of the theory at the level of the Ward identity which is equivalent to what is done in perturbation theory.

8.4 Path-integral approach

Using the formal path integrals of continuum theory in [13] the chiral anomaly has been claimed to arise from the measure. This can be checked with the proper definition of those integrals from the lattice. In Section 2.3 we have shown that alternative transformations allow to transport chirally noninvariant terms to the measure contribution. However, no such transformation has been used and no such term has been in the action in [13]. Further in Section 8.1 we have pointed out that the chiral transformation with (7.12) of the Atiyah-Singer case gives a measure contribution in (7.13). However, the transformation in [13] has not been of this type. Thus none of such mechanisms can be working there.

In [13] the result of the local chiral transformation is found to be ill-defined. The actual reason of this is that, instead of $\mathrm{Tr}(\gamma_5\hat{e}(n)) \equiv \mathrm{Tr}(\gamma_5 \otimes |n\rangle\langle n|)$ with proper vectors $|n\rangle$ which we have in (2.15), the formal continuum integrals there lead to $\sum_k \varphi_k(x)^\dagger\gamma_5\varphi_k(x) = \mathrm{Tr}(\gamma_5 \otimes |x\rangle\langle x|)$ with generalized vectors $|x\rangle$. The obvious problem then is that $|x\rangle\langle x|$ is no projection. This could be fixed by any of the known methods of dealing with generalized vectors. The most appropriate way, however, is to start properly from the discrete definition of the integrals in which case no such problem arises.

In the case of the global chiral transformation the proposal in [13] corresponds to the replacement of $\mathrm{Tr}\,\gamma_5$ by $\mathrm{Tr}(\gamma_5 f(D^2))$ with a function f which satisfies $f(0) = 1$ and has a suitable behavior at infinity. However, as discussed in Section 8.1 in this situation (8.1) and (8.2) apply and the result is zero not only in nonperturbative quantized theory but also from the point of view of the Atiyah-Singer theorem.

The desired result in [13] is nevertheless obtained essentially repeating the procedure of [11]. Instead of $f(D^2) = m^2(m^2-D^2)^{-1}$ there in [13] this is done with $f(D^2) = \exp(D^2/m^2)$ and the possible use of other functions is pointed out. As in [11] by keeping only the appropriate term the anomaly is obtained. Thus again the theory is modified at the level of the Ward identity which is equivalent to what is done in perturbation theory.

9. Conclusions

We have clarified the questions which have recently emerged comparing lattice results on chiral fermions with long-known continuum results on index relations, concerning restrictions found on the lattice which have no counterpart in the Atiyah-Singer framework. To settle such issues it has been necessary to care about mathematical details. We have started reconsidering finite-lattice results to get a safe basis, which has led to additional results. We then have addressed various aspects of the continuum limit, the results of which appear also useful in other contexts. This then has enabled us to perform the limit of the anomaly term reliably and to establish the index relation of the nonperturbative quantized theory. To compare it in detail with the Atiyah-Singer theorem we have derived corresponding relations in that case and worked out the differences. Finally we have analyzed conventional continuum approaches on the basis of our results.

We have taken advantage of the fact that it suffices to consider the theory in a background gauge field which can be done in a unitary space on the finite lattice and in Hilbert space in the limit. We have seen that with a background gauge field the relations of interest are the same ones for all expectation values. We have also given a family of alternative chiral transformations which lead to the same Ward identity as the usual one, however, allow to transport terms from the action contribution to the measure contribution.

On the finite lattice we have shown that to get rid of eigennilpotents as well as to obtain chirality of eigenprojections in addition to γ_5-hermiticity one needs normality of the Dirac operator D. With these properties we have got general rules for real modes. Applying them to the global chiral Ward identity we have obtained the sum rule of chiral differences. This rule, to allow for a nonvanishing index, has turned out to put severe restrictions on the spectrum of D. Using a particular decomposition of D we have studied spectral conditions and given a general expression for appropriate one-dimensional constraints.

Considering consequences for the GW relation we have shown that its general form does not guarantee normality of D and, in particular, that the index relation so far claimed for it does not hold. We have emphasized that one has to restrict to the simple form of the GW relation and that this form is actually a spectral constraint. We have stressed that with the alternative transformation which transports the anomaly term to the measure contribution – provided proper zero-mode regularization is used – the index term is still in the action contribution and thus the action noninvariant.

After specifying the continuum limit in detail we have studied the form of the spectral representations with the inclusion of a continuous spectrum of D and worked out the respective changes of the index relations. It has turned out that for a large class of spectra, which can be specified by appropriate constraints, still only the discrete spectrum contributes to these relations. Since the global chiral Ward identity is a particular decomposition of $\mathrm{Tr}\,\gamma_5 = 0$ we have established the validity of this trace relation and its connection to the cardinalities of the chiral subspaces in Hilbert space.

We have determined the conditions to be imposed on the lattice gauge fields in order that the correct limit exists. It has turned out that they, on the other hand, lead precisely to the continuity of the continuum fields which is relevant for topological properties. We have also cared about certain subtleties of the gauge-field limit by properly distinguishing representations and using the isomorphism between the respective spaces.

To determine the limit of the anomaly term satisfactorily we have used norm convergences in Hilbert space. The use of various site-diagonal operators has allowed us to get bounds conveniently. Further orderings of selfadjoint operators have repeatedly been useful to get lower bounds. Combining our results we then have obtained the index theorem as it follows in nonperturbative quantized theory.

To compare with the Atiyah-Singer index theorem we have derived corresponding relations in that case and then discussed the differences to nonperturbative quantized theory. We have stressed the fundamental one of different cardinalities of the chiral subspaces in the Atiah-Singer case and pointed out that the structure of the space itself there depends on the gauge-field configurations.

Finally we have analyzed conventional continuum approaches and shown that the Pauli-Villars approach as well as the path-integral approach neither conform with the nonperturbative quantized theory nor with the Atiyah-Singer framework but – as perturbation theory – rely on a modification of the theory at the level of the Ward identity.

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